ON THE UNIQUENESS OF GROUND STATES
OF NON-LOCAL EQUATIONS

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ABSTRACT. We review our joint result with E. Lenzmann about the uniqueness of ground state solutions of non-linear equations involving the fractional Laplacian and provide an alternate uniqueness proof for an equation related to the intermediate long-wave equation.

1. INTRODUCTION

We are interested in uniqueness of positive solutions Q of non-linear equations, of which

$$(-\Delta)^s Q - Q^{\alpha+1} = -Q \quad \text{in } \mathbb{R}^N \qquad (1.1)$$

with $0 < s < 1$ is a typical example. For s in this range, $(-\Delta)^s$ is a non-local operator. Throughout we will only be interested in finite energy solutions, i.e., $Q \in H^s(\mathbb{R}^N)$.

Before discussing (1.1) in more detail, we recall a few classical results about the local case with $s = 1$,

$$-\Delta Q - Q^{\alpha+1} = -Q \quad \text{in } \mathbb{R}^N. \qquad (1.2)$$

These results concern *existence*, *radial symmetry* and *uniqueness* of positive solutions. The existence of non-negative solutions for sub-critical values of α (i.e., $2 < \alpha + 2 < \infty$ if $N = 1, 2$ and $2 < \alpha + 2 < 2^* = 2N/(N-2)$ if $N \geq 3$) follows from the fact that the infimum

$$\inf_{u \in H^1(\mathbb{R}^N)} \frac{\left(\int |\nabla u|^2 \, dx\right)^\theta \left(\int |u|^2 \, dx\right)^{1-\theta}}{\left(\int |u|^{\alpha+2} \, dx\right)^{2/(\alpha+2)}}, \qquad \theta = \frac{N\alpha}{2(\alpha+2)},$$

is attained. Any minimizer, after scaling and multiplication by a constant, yields a non-negative solution to (1.2). By rearrangement inequalities this shows that (1.2) has a positive solution which is symmetric decreasing (i.e., radial with respect to some point and non-increasing with respect to the distance from this point). It is a theorem that *any* positive solution is (strictly) symmetric decreasing. This can be proved by the method of moving planes. The most delicate result about (1.2) is due to Kwong [17] (extending earlier results in [8, 23]) and states uniqueness of positive solutions vanishing at infinity. The proof is based on a careful ODE analysis. To summarize, for any subcritical α, (1.2) has a unique positive, finite-energy solution.

The question we would like to address here is whether the same is true for the non-local equation (1.1) with $0 < s < 1$. Equations of this type arise in numerous models from mathematical physics, mathematical biology and mathematical finance and we refer to [10] for precise references. Although (1.1) is time-independent, it plays a role in the description of traveling wave or solitary wave solutions of certain time-dependent equations.

Existence of positive solutions follows again by minimization for sub-critical values of α. Now 'sub-critical' means $2 < \alpha + 2 < \infty$ if $N \leq 2s$ and $2 < \alpha + 2 < 2N/(N-2s)$ if $N > 2s$. Radial symmetry and monotonicity of positive solutions has recently been established in [22] (see also [9]) by a modification of the method of moving planes. Thus, in what follows we shall concentrate on the *uniqueness* of positive, radial solutions of (1.1).

To understand why a uniqueness result is plausible and where the assumption $0 < s < 1$ comes from, we recall that the heat kernel of $(-\Delta)^s$, that is, the integral kernel of the operator $\exp(-t(-\Delta)^s)$, is positive precisely for $0 < s \leq 1$. By well-known arguments, this implies that positive solutions Q of the linear Schrödinger-type equation $(-\Delta)^s Q + VQ = -\mu Q$ are unique (up to a multiplicative constant). This suggests that the uniqueness property is closer related to the positivity of the heat kernel (and therefore to the maximum principle) than it is to locality and ODE techniques, which are crucial in Kwong's work [17].

The question of uniqueness has been raised for ground states of some specific non-local equations (see, e.g., [21] and the preprint version of [16]), but rigorous answers are rare in the literature. A celebrated result of Amick and Toland [5] concerns the case $N = 1$, $s = 1/2$ and $\alpha = 1$ of (1.1). Later, we will discuss in more detail a related result of Albert and Toland [4], again for $N = 1$ and $\alpha = 1$, but with a non-homogeneous operator. The only other case we are aware of concerns the critical case $\alpha + 2 = 2N/(N-2s)$ of (1.1) with the right side replaced by zero. Here *any* $0 < s < N/2$ is allowed, see [18, 7]. We emphasize that in all these cases the optimizers are known explicitly and the uniqueness proofs make use of these closed form expressions.

Recently, in [10] we proved a uniqueness result for energy-minimizing positive solutions of (1.1) in dimension $N = 1$. The parameter α is allowed to take any sub-critical value. The precise statement is the following.

Theorem 1.1. *Let $N = 1$ and assume that $0 < \alpha < \frac{4s}{1-2s}$ if $0 < s < \frac{1}{2}$ and that $0 < \alpha < \infty$ if $\frac{1}{2} \leq s < 1$. Then the infimum*

$$\inf_{u \in H^s(\mathbb{R}^N)} \frac{\|(-\Delta)^{s/2}u\|_2^\theta \, \|u\|_2^{1-\theta}}{\|u\|_{\alpha+2}}, \qquad \theta = \frac{\alpha}{2s(\alpha+2)},$$

is attained by a unique (modulo symmetries) function. More precisely, there is a positive, even and decreasing function Q such that any minimizer is of the form $aQ(b(x-c))$ for some $a \in \mathbb{C} \setminus \{0\}$, $b > 0$ and $c \in \mathbb{R}$.

Moreover, Q satisfies $(-\Delta)^s Q - Q^{\alpha+1} = -Q$, and the corresponding linearization

$$L_+ = (-\Delta)^s - (\alpha+1)Q^\alpha + 1 \qquad \text{in } L^2(\mathbb{R})$$

is non-degenerate, i.e., $\ker L_+ = \operatorname{span}\{Q'\}$.

Remarks. (1) The restriction on α are optimal for the existence of a minimizer.
(2) Most (but not all) of the arguments in our proof work for $N \geq 2$ as well.
(3) It is an *open problem* whether every positive solution is energy minimizing.
(4) No closed form expression for the functions Q are known unless $s = 1/2$ and $\alpha = 1$. Their regularity and (inverse power) decay are rather well understood.
(5) The fact that $Q' \in \ker L_+$ follows by differentiating the equation for Q. The difficult part is to prove that there is no linear independent element in $\ker L_+$. This so-called non-degeneracy result has applications in stability and blow-up analysis of non-linear dispersive equations; see, e.g., [16].

The idea behind our proof of Theorem 1.1 is to find a continuous branch of functions Q_s, $s \in [s_0, 1]$, such that Q_s solves (1.1) for any s and such that Q_{s_0} agrees with a given energy-minimizing solution of the equation. Uniqueness at s_0 is then deduced from uniqueness and non-degeneracy at $s = 1$, which are well known. The construction of the branch is based on the inverse function theorem, which requires one to check that a certain linear operator is invertible. To prove this invertibility we need a slightly stronger non-degeneracy statement than that in Theorem 1.1. A crucial ingredient of its proof is a result about the number of zeroes of an eigenfunction corresponding to the second eigenvalue of a Schrödinger-type operator $(-\Delta)^s + V$. It is only at this last point that we need the assumption that $N = 1$. For the details of the proof we refer the reader to [10].

Acknowledgment. I wish to thank E. Lenzmann in collaboration with whom Theorem 1.1 was obtained. His questions stimulated me to find the results presented in the following sections.

2. A uniqueness result for the ILW equation

We turn now to the uniqueness question for a one-dimensional equation which is slightly different from (1.1). It arises in the analysis of the intermediate long-wave (ILW) equation, which interpolates, in some sense, between the well-known KdV and Benjamin–Ono equations; see [15] and the references therein for background information. Travelling wave solutions of the ILW equation correspond to solutions $Q \in H^{1/2}(\mathbb{R})$ of the equation

$$TQ - Q^2 = -\mu Q \quad \text{in } \mathbb{R}. \tag{2.1}$$

Here T is the (non-local) pseudo-differential operator given by

$$\widehat{T\varphi}(\tau) = (\tau \coth(\pi\tau/2) - 2/\pi)\,\widehat{\varphi}(\tau)$$

in terms of the Fourier transform $\widehat{\varphi}(\tau) = (2\pi)^{-1/2} \int e^{-i\tau t}\varphi(t)\, dt$, and we are using dimensionless variables in which the height $H = \pi/2$. Our results concern the special case $\mu = 2/\pi$ of (2.1). We shall prove

Theorem 2.1. *For any $\varphi \in H^{1/2}(\mathbb{R})$ one has*

$$(\varphi, T\varphi) + \tfrac{2}{\pi}\|\varphi\|_2^2 \geq \left(\tfrac{\pi}{2}\right)^{1/3}\|\varphi\|_3^2$$

with equality iff $\varphi(x) = \beta\, h(t - \alpha)$ for some $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$, where

$$h(t) = \frac{1}{\cosh t}\,.$$

This result implies the existence of a positive solution to equation (2.1) with $\mu = 2/\pi$ and states that it is unique (up to translations) among energy-minimizing solutions. The next result shows that, actually, the energy-minimizing property is not necessary for this uniqueness result. In particular, any positive solution is automatically energy-minimizing.

Theorem 2.2. *Assume $Q \in H^{1/2}(\mathbb{R})$ is non-negative and satisfies*

$$TQ - Q^2 = -\tfrac{2}{\pi}Q \qquad in\ \mathbb{R}\,. \tag{2.2}$$

Then $Q(x) = h(t - \alpha)$ for some $\alpha \in \mathbb{R}$. Moreover,

$$\ker\left(T - 2Q + \tfrac{2}{\pi}\right) = \mathrm{span}\{Q'\}\,.$$

Remarks. (1) The first part of this theorem is due to Albert and Toland [4] (see also [1]) and the second part due to Albert and Bona [2]. They can treat equation (2.1) with general $\mu > 0$.
(2) The assumption $Q \geq 0$ in Theorem 2.2 can be replaced by the weaker assumption that Q is real-valued. To see this, write the equation as $Q = (T + \tfrac{2}{\pi})^{-1}Q^2$ and use the fact that $(T + \mu)^{-1}$ has a positive integral kernel for any $\mu > 0$; see, e.g., [1].
(3) The fact that $Q' \in \ker\left(T - 2Q + \tfrac{2}{\pi}\right)$ follows from translation-invariance of (2.2). The non-obvious fact is that there is no linearly independent function in this kernel.
(4) Equation (2.2) is understood in $H^{-1/2}(\mathbb{R})$, that is, for any $\varphi \in H^{1/2}(\mathbb{R})$

$$\int_{\mathbb{R}} \tau \coth(\pi\tau/2)\, \overline{\widehat{\varphi}(\tau)}\widehat{Q}(\tau)\, d\tau = \int_{\mathbb{R}} \overline{\varphi(t)}Q(t)^2\, dt\,.$$

The reason why we think our new proofs of Theorems 2.1 and 2.2 may be worth recording is that they establish a connection between equation (2.2) and the seemingly unrelated equation

$$\sqrt{-\Delta}R = R^2 \quad in\ \mathbb{R}^3\,.$$

This will allow us to deduce Theorems 2.1 and 2.2 from known results about the latter equation. Here is our key lemma.

Lemma 2.3. *Let φ be a function on $\mathbb{R}$ and ψ be a radial function on $\mathbb{R}^3$ related by*

$$\psi(x) = |x|^{-1}\varphi(\ln|x|)\,.$$

Then $\varphi \in H^{1/2}(\mathbb{R})$ iff $\psi \in \dot{H}^{1/2}(\mathbb{R}^3)$ and, in this case,

$$(\psi, \sqrt{-\Delta}\psi) = 4\pi \int_{\mathbb{R}} \tau \coth(\pi\tau/2)|\widehat{\varphi}(\tau)|^2 \, d\tau\,.$$

By way of comparison, we note that if $\psi(x) = |x|^{-1/2}\varphi(\ln|x|)$, then

$$\int_{\mathbb{R}^3} |\nabla\psi|^2 \, dx = 4\pi \int_{\mathbb{R}} \left(|\varphi'|^2 + \tfrac{1}{4}|\varphi|^2\right) \, dt\,.$$

Actually, Lemma 2.3 works both ways: Not only can we deduce Theorems 2.1 and 2.2 from three-dimensional results, we also obtain sharp functional inequalities in three dimensions from the one-dimensional Albert–Toland result [4].

Theorem 2.4. *For any $\psi \in H^{1/2}(\mathbb{R}^3)$ and any $\theta \in (0, \pi/2)$ one has*

$$(\psi, \sqrt{-\Delta}\psi) - \tfrac{2}{\pi}\theta\cot\theta\,(\psi, |x|^{-1}\psi) \geq \left(\frac{4\theta}{\sqrt{\pi}\sin\theta}\right)^{\frac{2}{3}} \left(\theta\,(2 + \cos(2\theta)) - \tfrac{3}{2}\sin(2\theta)\right)^{\frac{1}{3}} \|\psi\|_3^2\,,$$

with equality iff $\psi(x) = cH_\theta(bx)$ for some $b > 0$ and $c \in \mathbb{C}$, where

$$H_\theta(x) = |x|^{-1}\left(|x|^{2\theta/\pi} + |x|^{-2\theta/\pi} + 2\cos\theta\right)^{-1}\,.$$

We note that $\theta\cot\theta$ strictly decreases as function of θ from 1 at $\theta = 0$ to 0 at $\theta = \pi/2$. Thus Theorem 2.4 interpolates between Kato's inequality $(\psi, \sqrt{-\Delta}\psi) \geq (2/\pi)(\psi, |x|^{-1}\psi)$ and the Sobolev inequality $(\psi, \sqrt{-\Delta}\psi) \geq 2^{1/3}\pi^{2/3}\|\psi\|_3^2$ (see Lemma 2.1 below).

3. Proofs

3.1. **Proof of the key lemma.** The ground state substitution formula from [12] (see also [13]) reads

$$(\psi, \sqrt{-\Delta}\psi) - \frac{2}{\pi}\int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|}dx = \frac{1}{2\pi^2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\chi(x) - \chi(y)|^2}{|x-y|^4}\frac{dx\,dy}{|x|\,|y|}\,,$$

where $\psi(x) = |x|^{-1}\chi(x)$. This is true for any $\psi \in \dot{H}^{1/2}(\mathbb{R}^3)$. Now assume that ψ is radial. Then, using the fact that for $|x| = r$

$$\int_{\mathbb{S}^2} \frac{d\omega}{|x - s\omega|^4} = 2\pi\int_0^\pi \frac{\sin\theta d\theta}{(r^2 - 2rs\cos\theta + s^2)^2} = 2\pi\int_{-1}^1 \frac{dt}{(r^2 - 2rst + s^2)^2} = \frac{4\pi}{(r^2 - s^2)^2}\,,$$

we obtain

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\chi(x) - \chi(y)|^2}{|x-y|^4}\frac{dx\,dy}{|x|\,|y|} = (4\pi)^2 \iint_{\mathbb{R}_+\times\mathbb{R}_+} \frac{|\chi(r) - \chi(s)|^2}{(r^2 - s^2)^2}rs\,dr\,ds\,.$$

Next, we change variables $r = e^t$ and $s = e^u$. Writing $\chi(r) = \varphi(\ln r)$ we find

$$\iint_{\mathbb{R}_+ \times \mathbb{R}_+} \frac{|\chi(r) - \chi(s)|^2}{(r^2 - s^2)^2} rs \, dr \, ds = \iint_{\mathbb{R} \times \mathbb{R}} |\varphi(t) - \varphi(u)|^2 \frac{e^{2(t+u)}}{(e^{2t} - e^{2u})^2} \, dt \, du$$

$$= \frac{1}{4} \iint_{\mathbb{R} \times \mathbb{R}} \frac{|\varphi(t) - \varphi(u)|^2}{\sinh^2(t-u)} \, dt \, du \,.$$

Summarizing what we have shown so far is that

$$(\psi, \sqrt{-\Delta}\psi) = 2 \iint_{\mathbb{R} \times \mathbb{R}} \frac{|\varphi(t) - \varphi(u)|^2}{\sinh^2(t-u)} \, dt \, du + 8 \int_{\mathbb{R}} |\varphi(t)|^2 \, dt \,.$$

Now we take the Fourier transform and find

$$\iint_{\mathbb{R} \times \mathbb{R}} \frac{|\varphi(t) - \varphi(u)|^2}{\sinh^2(t-u)} \, dt \, du = 4 \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\sin^2(\tau t/2)}{\sinh^2 t} \, dt \right) |\widehat{\varphi}(\tau)|^2 \, d\tau$$

$$= 8\pi \int_{\mathbb{R}} \left(\frac{\tau \coth(\pi\tau/2)}{4} - \frac{1}{2\pi} \right) |\widehat{\varphi}(\tau)|^2 \, d\tau \,, \qquad (3.1)$$

where the last equality used [14, (3.986.4)]. This proves Lemma 2.3. $\qquad\square$

3.2. **Proofs of Theorems 2.1 and 2.2.** Given our Key Lemma 2.3, we can deduce Theorems 2.1 and 2.2 from two known results in $\mathbb{R}^3$. The first is the following sharp Sobolev inequality due, in a dual form, to Lieb [19]; see also [20, Thm. 8.4].

Theorem 3.1. *For any $\psi \in \dot{H}^{1/2}(\mathbb{R}^3)$ one has*

$$(\psi, \sqrt{-\Delta}\psi) \geq 2^{1/3}\pi^{2/3}\|\psi\|_3^2 \,, \qquad (3.2)$$

with equality iff $\psi(x) = cH(b(x-a))$ for some $a \in \mathbb{R}^3$, $b > 0$ and $c \in \mathbb{C}$, where

$$H(x) = (1 + |x|^2)^{-1} \,. \qquad (3.3)$$

If we combine this theorem with Lemma 2.3, we obtain

$$4\pi \int_{\mathbb{R}} \tau \coth(\pi\tau/2)|\widehat{\varphi}(\tau)|^2 \, d\tau = (\psi, \sqrt{-\Delta}\psi)$$

$$\geq 2^{1/3}\pi^{2/3} \left(\int_{\mathbb{R}^3} |\psi|^3 \, dx \right)^{2/3} = 2^{5/3}\pi^{4/3} \left(\int_{\mathbb{R}} |\varphi|^3 \, dt \right)^{2/3} \,,$$

with equality iff $\psi(x) = c \, H(bx)$ for some $b > 0$ and $c \in \mathbb{C}$, that is,

$$\varphi(t) = \frac{c}{2b \, \cosh(t + \ln b)}$$

This proves Theorem 2.1.

For the proof of Theorem 2.2 we need the following result of Li [18] and Chen–Li–Ou [7], which had been conjectured by Lieb [19]. Since we have not been able to find a proof of the non-degeneracy statement in the literature we provide one in the appendix.

Theorem 3.2. *Assume that $R \in \dot{H}^{1/2}(\mathbb{R}^3)$ is non-negative and satisfies*

$$\sqrt{-\Delta}R = R^2 \qquad in \ \mathbb{R}^3 \, . \tag{3.4}$$

Then $R(x) = 2b\, H(b(x-a))$ for some $a \in \mathbb{R}^3$ and $b \geq 0$ with H from (3.3). Moreover,

$$\ker\left(\sqrt{-\Delta} - 2R\right) = \operatorname{span}\{\partial_1 R, \partial_2 R, \partial_3 R, R + x \cdot \nabla R\} \, . \tag{3.5}$$

The proof of Theorem 2.2 is now similar to that of Theorem 2.1. Indeed, polarization of the identity in Lemma 2.3 shows that equation (2.2) for Q becomes equation (3.4) for $R(x) = |x|^{-1}Q(\ln|x|)$. Thus, by Theorem 3.2, $R(x) = 2bH(bx)$, that is, $Q(t) = (\cosh(t + \ln b))^{-1}$, as claimed.

To prove the non-degeneracy statement for Q we use the fact that, since R is centered at the origin, the operator $\sqrt{-\Delta} - 2R$ commutes with rotations about the origin. Therefore the subspace of radial functions in $L^2(\mathbb{R}^3)$ is a reducing subspace for this operator and, according to (3.5), its kernel restricted to this subspace is spanned by $R + x \cdot \nabla R$. On the other hand, Lemma 2.3 implies that there is a one-to-one correspondence between the kernel of $T - 2Q + \frac{2}{\pi}$ and the kernel of $\sqrt{-\Delta} - 2R$ on radial functions. The assertion now follows from the fact that $R + x \cdot \nabla R = |x|^{-1}Q'(\ln|x|)$ if $R(x) = |x|^{-1}Q(\ln|x|)$. This completes the proof of Theorem 2.2.

3.3. **Proof of Theorem 2.4.** By rearrangement inequalities [20, Thm. 3.4 and Lemma 7.17] (which are *strict* for the $(\psi, |x|^{-1}\psi)$-term), we only need to consider radial functions. On such functions, the theorem is equivalent to

$$(\varphi, T\varphi) + \tfrac{2}{\pi}\left(1 - \theta\cot\theta\right)\|\varphi\|_2^2 \geq \left(\frac{2\theta}{\pi\sin\theta}\right)^{2/3}\left(\theta\left(2 + \cos(2\theta)\right) - \tfrac{3}{2}\sin(2\theta)\right)^{1/3}\|\varphi\|_3^2, \tag{3.6}$$

with equality iff $\varphi(t) = \beta h_\theta(t - \alpha)$ for some $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$, where

$$h_\theta(t) = \frac{(2\theta/\pi)\sin\theta}{\cosh(2\theta t/\pi) + \cos\theta} \, .$$

This follows from [4]. Indeed, standard arguments (see, e.g., [3]) yield that the infimum

$$I_\theta = \inf_{\varphi \in H^{1/2}(\mathbb{R})} \frac{(\varphi, T\varphi) + \tfrac{2}{\pi}\left(1 - \theta\cot\theta\right)\|\varphi\|_2^2}{\|\varphi\|_3^2}$$

is attained by a function φ_θ. Using (3.1) one can show that any minimizer is a multiple of a non-negative function, and therefore we may assume that $\varphi_\theta \geq 0$. The Euler–Lagrange equation reads

$$T\varphi_\theta + \tfrac{2}{\pi}\left(1 - \theta\cot\theta\right)\varphi_\theta = \lambda\varphi_\theta^2$$

with a Lagrange multiplier λ. Integration against φ_θ shows that $I_\theta = \lambda\|\varphi_\theta\|_3$. The function $Q = \lambda\varphi_\theta$ satisfies (2.1) and therefore, by [4], $Q(t) = h_\theta(t)$. Finally, by a

computation,

$$I_\theta = \|Q\|_3 = \frac{2\theta \sin\theta}{\pi} \left(2 \int_0^\infty \frac{dt}{(\cosh(2\theta t/\pi) + \cos\theta)^3}\right)^{1/3}$$

$$= \left(\frac{2\theta}{\pi \sin\theta}\right)^{2/3} \left(\theta\left(2 + \cos(2\theta)\right) - \tfrac{3}{2}\sin(2\theta)\right)^{1/3},$$

which concludes the proof of Theorem 2.4.

Remark. Inequality (3.6) is true for all $\theta \in (0, \pi)$ (by the same proof) and therefore the inequality in Theorem 2.4 is true in this range *for radial functions* ψ. However, rearrangement inequalities for the $(\psi, |x|^{-1}\psi)$-term work in the wrong way for $\theta \in (\pi/2, \pi)$.

Appendix A. Non-degeneracy for $\sqrt{-\Delta}$ in three dimensions

In this appendix we prove (3.5). Our argument is based on Lieb's observation [18] that inequality (3.2), and therefore also equation (3.4), are conformally invariant. To utilize this fact, we first turn the differential equation into an integral equation by noting that

$$\ker\left(\sqrt{-\Delta} - 2R\right) = \ker\left(2R^{1/2}(-\Delta)^{-1/2}R^{1/2} - 1\right).$$

Indeed, $\sqrt{-\Delta}\psi = 2R\psi$ iff $2R^{1/2}(-\Delta)^{-1/2}R^{1/2}\chi = \chi$, where $\chi = R^{1/2}\psi$. (In mathematical physics, this is called the *Birman–Schwinger principle*.) The integral kernel of the operator $R^{1/2}(-\Delta)^{-1/2}R^{1/2}$ appearing on the right side is

$$(2\pi^2)^{-1}\sqrt{R(x)}|x - y|^{-2}\sqrt{R(y)}. \tag{A.1}$$

To proceed, we note that we may assume that $a = 0$ and $b = 1$ in the formula for R. We introduce the stereographic projection $\mathcal{S} : \mathbb{R}^3 \to \mathbb{S}^3 = \{(\omega_1, \ldots, \omega_4) \in \mathbb{R}^4 : \sum \omega_j^2 = 1\}$,

$$\mathcal{S}_j(x) = \frac{2x_j}{1 + x^2} \quad \text{for } j = 1, \ldots, 3, \qquad \mathcal{S}_4(x) = \frac{1 - x^2}{1 + x^2}.$$

The Jacobian of $\mathcal{S}$ is

$$\mathcal{J}(x) = \left(\frac{2}{1 + x^2}\right)^3,$$

and therefore, we obtain a unitary operator $\mathcal{U} : L^2(\mathbb{S}^3) \to L^2(\mathbb{R}^3)$ by setting

$$\mathcal{U}\Psi(x) = \mathcal{J}(x)^{1/2}\Psi(\mathcal{S}(x)), \tag{A.2}$$

for $\Psi \in L^2(\mathbb{S}^3)$. Using the fact that

$$|\mathcal{S}(x) - \mathcal{S}(y)|^2 = \mathcal{J}(x)|x - y|^2\mathcal{J}(y),$$

we find that the operator $K = \mathcal{U}^* R^{1/2}(-\Delta)^{-1/2}R^{1/2}\mathcal{U}$ has kernel

$$K(\omega, \eta) = (2\pi^2)^{-1}|\omega - \eta|^{-2}.$$

Since this operator commutes with rotations of $\mathbb{S}^3$, its eigenfunctions are spherical harmonics and its eigenvalues can be computed explicitly using the Funk–Hecke formula, see [6] and also [11]. The eigenvalues are given by $(l+1)^{-1}$, $l \in \mathbb{N}_0$, and the multiplicity of the l-th eigenvalue is $(l+1)^2$. Here l is the degree of the spherical harmonic. We conclude that

$$\ker \mathcal{U}^* \left(2R^{1/2}(-\Delta)^{-1/2}R^{1/2} - 1 \right) \mathcal{U} = \ker\left(2K - 1\right)$$

corresponds to $l = 1$ and is therefore spanned by the functions $\Psi_j(\omega) = \omega_j$, $j = 1, \ldots, 4$. Thus $\ker\left(2R^{1/2}(-\Delta)^{-1/2}R^{1/2} - 1\right)$ is spanned by the $\mathcal{U}\Psi_j = \mathcal{J}^{1/2}\mathcal{S}_j = R^{3/2}\mathcal{S}_j$ and, again by the Birman–Schwinger principle, $\ker\left(\sqrt{-\Delta} - 2R\right)$ is spanned by the $R\mathcal{S}_j$. These are the claimed functions and our proof of (3.5) is complete.

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